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NEW DEVELOPMENTS CONCERNING PIEZOELECTRIC MATERIALS WITH DEFECTS

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Abstract—The problem of a piezoelectric body with an elliptic cavity is revisited within the framework of two-dimensional electro-elasticity. In contrast to our previous formulation, the present analysis is based on the use of exact electric boundary conditions at the rim of the hole, thus avoiding the common assumption of electric impermeability. Expressions for the elastic and electric variables induced inside and outside the cavity are derived in closed form in terms of complex potentials. Comparisons between present (exact) and previous (approximate) models are effected to establish the limitations associated with the assumption of the impermeable hole, in particular when it becomes a slit crack. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

The theme of the elliptic hole (and, in the limit, of the crack) embedded in an infinite body made of piezoelectric material has been addressed by many authors in the recent past. The problem has generated a wealth of different approaches and results, which for the most part are still awaiting experimental verification. The topic has a motivation which is remote from being purely academic. Indeed, the use of piezoelectric polymers and ceramics in modern technology has induced researchers to investigate the mechanics of deformation and failure of these materials. Two typical examples of applications are furnished by the areas of electronic packaging and intelligent structures.

Phenomenological descriptions of piezoelectric solids with defects such as holes, inclusions and cracks are mathematically difficult to develop due to two factors: electroelastic coupling effects and material anisotropy. Thus, in an effort to minimize such difficulties, many authors have proposed alternative models based on assumptions regarding defect orientation, nature of the applied loads and associated deformations and form of the boundary conditions imposed at the defect's surface. Examples on this line of work are the articles of Pak (1990, 1992a, 1992b) within the realm of anti-plane piezoelectricity, and the more general studies of Deeg (1980), Sosa and Pak (1990), Shindo *et al.* (1990) Sosa (1991, 1992) and Suo *et al.* (1992).

Because of their practical relevance, the articles mentioned above were mainly concerned with transversely isotropic piezoelectric materials like poled ferroelectric ceramics and crystals of the 6mm class. In addition, they were built upon the assumption of a void (or crack) filled with gas (typically air or vacuum) and with its boundary free of forces and electric surface charge. In electrostatics, at a surface separating two dielectric mediums, the electric potential (or, alternatively, the tangential component of the electric field) and the normal component of the induction vector are continuous. However, when one of the mediums is air, these two conditions can be approximated simply by one, namely that the normal component of the induction, as calculated in the other medium, vanishes at the interface.* Regarding the boundary of a hole or crack as a surface separating non conducting matter from air, the aforementioned authors adhered to the impermeable assumption due to its much simpler mathematical treatment. However, the approximation has a drawback: when the defect is a slit crack the electric field becomes singular at its tip.

* Hereafter such approximation will be referred to as the *impermeable* approximation.

First McMeeking (1989), within the framework of isotropic electrostriction, and recently Dunn (1994), in terms of anti-plane piezoelectricity, have pointed out that the singular behavior of the electric field at the tip of a sharp crack is an anomaly caused by the impermeable assumption. In fact, they have found that the magnitude of the field remains bounded by the permitivities of the piezoelectric material and gas enclosed by the crack. The present article is devoted to a generalization of the results presented in these two works by including material anisotropy (absent in the case of electrostriction) in a plane where full electromechanical coupling takes place (a phenomenon which does not manifest in anti-plane deformations). Therefore, we are revisiting the plane problem of Sosa (1991) rejecting in this case the impermeable approximation. In the process, first, we recover the results of McMeeking and Dunn; second, we show that the impermeable formulation can be derived as a particular case of the exact model; and third we make pertinent comparisons between both formulations.

The solutions here provided, together with those available for anti-plane deformations seem to close the topic of a defect in a piezoelectric solid from a two-dimensional point of view. More complete formulations (such as three-dimensional configurations or cracks with arbitrary directions) have also been provided by some authors. In these cases, however, the solutions do not have simple forms and tend to hinder the physical aspects involved in the mechanical and electric failure of piezoelectric materials. It is probably at this stage where numerical approaches become the only reasonable alternative.

2. REVIEW OF BASIC EQUATIONS

As in Sosa (1991) (hereafter referred to as HS) we effect a plane strain analysis of transversely isotropic bodies referred to a Cartesian coordinate system x_1, x_2. Denoting by* S_{ij}, T_{ij}, D_i and E_i the components of strain, stress, induction and electric field, respectively, the two-dimensional constitutive equations can be written in matrix form as

$$\begin{Bmatrix} S_{11} \\ S_{22} \\ 2S_{12} \end{Bmatrix} = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{Vmatrix} \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{Bmatrix} + \begin{Vmatrix} 0 & b_{21} \\ 0 & b_{22} \\ b_{13} & 0 \end{Vmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{1a}$$

$$\begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\begin{Vmatrix} 0 & 0 & b_{13} \\ b_{21} & b_{22} & 0 \end{Vmatrix} \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{Bmatrix} + \begin{Vmatrix} c_{11} & 0 \\ 0 & c_{22} \end{Vmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{1b}$$

where a_{ij}, b_{ij} and c_{ii} are the components of the reduced (or effective) elastic compliance, piezoelectric and dielectric impermittivity (or impermeability) matrices, respectively.

We have shown in HS that mechanical and electrical fields arising in piezoelectric bodies can be expressed in terms of three complex potentials φ_k (as functions of three complex variables $z_k = x_1 + \mu_k x_2$) and their corresponding derivatives $\varphi_k' = \mathrm{d}\varphi_k/\mathrm{d}z_k$. For example:

$$\mathbf{T} = 2\mathscr{R} \sum_{k=1}^{3} \begin{Bmatrix} \mu_k^2 \\ 1 \\ -\mu_k \end{Bmatrix} \varphi_k'; \quad \mathbf{D} = 2\mathscr{R} \sum_{k=1}^{3} \begin{Bmatrix} \mu_k \\ -1 \end{Bmatrix} \lambda_k \varphi_k';$$

$$\mathbf{E} = 2\mathscr{R} \sum_{k=1}^{3} \begin{Bmatrix} 1 \\ \mu_k \end{Bmatrix} \kappa_k \varphi_k'; \quad \phi = -2\mathscr{R} \sum_{k=1}^{3} \kappa_k \varphi_k \tag{2a–d}$$

where **T**, **D** and **E** are the column vectors appearing in (1a–b), $\mathscr{R}\{.\}$ denotes the real part

* The reader is referred to section 2 of HS for a detailed account on the derivation of the relations given in this section with the caution that some change in notation has taken place to follow standard texts on piezoelectricity like IEEE (1988) and Ikeda (1990).

of the quantity within braces, and μ_k are three complex roots (with positive imaginary parts) satisfying the characteristic equation

$$a_{11}c_{11}\mu^6+[a_{11}c_{22}+2a_{12}c_{11}+a_{33}c_{11}+b_{21}^2+b_{13}^2+2b_{21}b_{13}]\mu^4+[a_{22}c_{11}+2a_{12}c_{22}+a_{33}c_{22} +2b_{21}b_{22}+2b_{13}b_{22}]\mu^2+a_{22}c_{22}+b_{22}^2=0$$

Furthermore, in (2b–d)

$$\lambda_k = -\frac{(b_{21}+b_{13})\mu_k^2+b_{22}}{c_{11}\mu_k^2+c_{22}} \tag{3}$$

$$\kappa_k = (b_{13}+c_{11}\lambda_k)\mu_k \tag{4}$$

If the body does not exhibit piezoelectricity (or this is neglected) $b_{ij}=0$, in which case $\lambda_k=\mu_k=0$ and the characteristic equation can be factored out into fourth and second degree equations to find the roots associated with problems of elasticity and electrostatics. It is also interesting to note that transversely isotropic bodies have such material symmetry that: $\mathscr{R}\{\mu_1\}=\mathscr{I}\{\lambda_1\}=\mathscr{R}\{\kappa_1\}=0$, $\mu_3=-\bar{\mu}_2$, $\lambda_3=\bar{\lambda}_2$ and $\kappa_3=-\bar{\kappa}_2$, where $\mathscr{I}\{\cdot\}$ stands for the imaginary part of the quantity enclosed in braces and the overbar indicates the complex conjugate.

The determination of the complex potentials is constrained to boundary conditions of mechanical and electrical type, which are discussed in the sequel within the realm of the problem addressed in this article. Towards this end let the infinite piezoelectric body contain an elliptic hole (of contour denoted by Γ and outward unit normal $\mathbf{n}$) with major and minor axes $2a$ and $2b$ along x_1 and x_2, respectively. The cavity is assumed to be filled with a homogeneous gas* of dielectric constant (or permittivity) ε_0, and is free of forces and surface charge density. On the other hand, mechanical and electrical loads in the form of forces, displacements, charge or voltages are applied to the body at remote distances. The main purpose of the paper is to find expressions for the elastic and electric variables both in the body and cavity, regions that we shall denote by Ω and Ω_c, respectively. In the latter, only electric variables $\mathbf{D}^c$ and $\mathbf{E}^c=-\nabla\phi^c$ exist, which are found from the solution of Laplace's equation for the electric potential ϕ^c, that is

$$\nabla^2\phi^c = 0 \quad \text{in}\,\Omega_c \tag{5}$$

In turn, the electric displacement is found via the constitutive relation

$$\mathbf{D}^c = \varepsilon_0\mathbf{E}^c \tag{6}$$

and its normal component is given by

$$\mathbf{D}^c\cdot\mathbf{n} = -\varepsilon_0\frac{\partial\phi^c}{\partial n} \tag{7}$$

The boundary conditions must state that the hole is traction free and that on its contour the normal component of the induction and the electric potential are continuous. Thus

$$t_1=0;\quad t_2=0;$$

$$D_n = -\varepsilon_0\frac{\partial\phi^c}{\partial n};\quad \text{on }\Gamma$$

* In most cases we can consider that the gas is simply air or perhaps vacuum in which case $\varepsilon_0 = 8.85\times10^{-12}\,\text{N/V}^2$.

$$\phi = \phi^c \qquad (8a–d)$$

where t_1 and t_2 are the Cartesian components of the stress vector. In HS, conditions (8c) and (8d) were approximated by the single equation $D_n = 0$ (the so-called condition of impermeability), based on the fact that the dielectric constants of a piezoelectric material are much larger than the one of vacuum (in fact, for most ceramics they can be between 1000 and 3500 times larger). The point of departure in the present article is that such simplification is not considered, and (8) is taken as such. We shall show, however, that by setting $\varepsilon_0 = 0$ in the expressions to be derived we can recover the results deduced in HS.

The analysis to follow requires the boundary conditions to be expressed in terms of the complex potentials. We can show that (8) becomes

$$2\sum_{k=1}^{3}\mathscr{R}\{\varphi_k\} = 0;\quad 2\sum_{k=1}^{3}\mathscr{R}\{\mu_k\varphi_k\} = 0;$$

$$2\sum_{k=1}^{3}\mathscr{R}\{\lambda_k\varphi_k\} = \int_0^s D_n\,\mathrm{d}s = -\varepsilon_0\frac{\partial\phi^c}{\partial n};$$

$$2\sum_{k=1}^{3}\mathscr{R}\{\kappa_k\varphi_k\} = -\phi^c \qquad (9a–d)$$

where ds is an element of arc length of Γ.

3. A SOLUTION THROUGH CONFORMAL MAPPING

In our previous article (HS), the assumption of dealing with a hole impermeable to electric fields had, as a consequence, the need for modelling only the region occupied by matter. In the present work, the cavity is also modelled, which means that we are confronted with a two-domain boundary-value problem much more difficult to solve in closed form than the one addressed in HS. Again, conformal mapping is the fundamental tool used to find the complex potentials. The region Ω (in the z-plane) is mapped onto the outside of the unit circle (in the ζ-plane) by means of the function

$$z_k = \omega_k(\zeta_k) = \frac{a-i\mu_k b}{2}\zeta_k + \frac{a+i\mu_k b}{2}\frac{1}{\zeta_k} \qquad (10)$$

which actually corresponds to three mappings, one for each root μ_k. The mapping of Ω_c is done by considering a straight line Γ_0 along x_1 and of length $2\sqrt{a^2-b^2}$. The region enclosed by the ellipse excluding the line Γ_0 can be mapped onto the ζ-plane by the function

$$z = \omega(\zeta) = \frac{a+b}{2}\zeta + \frac{a-b}{2}\frac{1}{\zeta} \qquad (11)$$

by means of which Γ and Γ_0 transform into the ring of outer and inner boundaries γ and γ_0, respectively, with parametric representations

$$\gamma_0 : \zeta = \rho_0 e^{i\theta};\quad \gamma : \zeta = \rho e^{i\theta} = e^{i\theta} = \sigma$$

where θ is the angle measured over the circles in counter clockwise direction and the radii have the values

$$\rho_0 = \sqrt{\frac{a-b}{a+b}}; \quad \rho = 1$$

In turn, the boundary conditions must be expressed over the unit circle. As we shall see in the next section, the left hand sides of (9) can be written in terms of holomorphic functions, while evaluation of the right hand sides in the transformed domain is carried out in the remaining part of this section.

To this end, we start by considering the electric potential inside the cavity, which can be expressed as

$$\phi^c(x_1, x_2) = F(z) + \overline{F(z)} \tag{12}$$

where the function $F(z)$ is analytic in the region between Γ_0 and Γ, and which in the ζ-plane becomes

$$F(z) = \Phi(\zeta) \tag{13}$$

Moreover, anywhere inside the hole and along the line Γ_0 the following condition must be satisfied:

$$\Phi(\rho_0 e^{i\theta}) = \Phi(\rho_0 e^{-i\theta}) \tag{14}$$

to ensure the field is single valued. Next, we consider the normal component of the induction vector, which in terms of the electric potential is given by

$$D_n\,\mathrm{d}s = -\varepsilon_0\left[\frac{\partial\phi^c}{\partial z}n + \frac{\partial\phi^c}{\partial\bar{z}}\bar{n}\right]\mathrm{d}s = -\varepsilon_0[f'(z)n + \overline{f'(z)}\bar{n}]\,\mathrm{d}s \tag{15}$$

Furthermore, since, for any arbitrary circle of radius ρ

$$z = \omega(\zeta) = \omega(\rho e^{i\theta}) \tag{16}$$

the following two conditions are derived for the normal and arc length of the curve Γ:

$$n = n_1 + in_2 = \frac{\mathrm{d}z(\rho)}{|\mathrm{d}z(\rho)|} = \frac{\omega'(\zeta)}{|\omega'(\zeta)|}e^{i\theta} \tag{17}$$

and

$$\mathrm{d}s(\theta) = |\mathrm{d}z(\theta)| = |\omega'(\zeta)|\rho\,\mathrm{d}\theta \tag{18}$$

Using the relation

$$F'(z) = \frac{\Phi'(\zeta)}{\omega'(\zeta)}$$

together with (15) and (18) when $\rho = 1$ yields

$$D_n\,\mathrm{d}s = -\varepsilon_0[\zeta\Phi'(\zeta) + \overline{\zeta\Phi'(\zeta)}]\,\mathrm{d}\theta$$

whose evaluation over the unit circle gives

$$\int_0^s D_n \, ds = -\frac{\varepsilon_0}{i}\{[\Phi(\sigma)-\Phi(1)]-[\overline{\Phi(\sigma)}-\overline{\Phi(1)}]\}. \tag{19}$$

We note that the constants $\Phi(1)$ and $\overline{\Phi(1)}$ will be omitted from the rest of the discussion since they do not contribute to either the components of stress or electric field.

4. DERIVATION OF THE COMPLEX POTENTIALS

The complex potentials are constructed by means of the series method. Towards this end let them be expressed as*

$$\varphi_k(z_k) = c_k z_k + \Phi_k^0(\zeta_k), \quad \text{for } |\zeta| \geqslant 1 \tag{20}$$

where

$$\Phi_k^0(\zeta_k) = a_{k0} + \sum_{j=1}^{\infty} \frac{a_{kj}}{\zeta_k^j} \tag{21}$$

are holomorphic functions up to infinity. Furthermore, let

$$\Phi(\zeta) = \sum_{j=1}^{\infty} \frac{d_{-j}}{\zeta^j} + d_0 + \sum_{j=1}^{\infty} d_j \zeta^j, \quad \text{for } \rho_0 \leqslant |\zeta| \leqslant 1 \tag{22}$$

whose coefficients can be related by means of (14) in the following manner:

$$d_{-j} = \rho_0^{2j} d_j \tag{23}$$

Substituting (10) and (20) in the left-hand side of (9) yields the boundary conditions on the unit circle, namely

$$\sum_{k=1}^{3} \Phi_k^0(\sigma) + \overline{\Phi_k^0(\sigma)} = \bar{l}_1 \sigma + \frac{l_1}{\sigma}$$

$$\sum_{k=1}^{3} \mu_k \Phi_k^0(\sigma) + \bar{\mu}_k \overline{\Phi_k^0(\sigma)} = \bar{l}_2 \sigma + \frac{l_2}{\sigma}$$

$$\sum_{k=1}^{3} \lambda_k \Phi_k^0(\sigma) + \bar{\lambda}_k \overline{\Phi_k^0(\sigma)} = \bar{l}_3 \sigma + \frac{l_3}{\sigma} - \frac{\varepsilon_0}{i}[\Phi(\sigma) - \overline{\Phi(\sigma)}]$$

$$\sum_{k=1}^{3} \kappa_k \Phi_k^0(\sigma) + \bar{\kappa}_k \overline{\Phi_k^0(\sigma)} = \bar{l}_4 \sigma + \frac{l_4}{\sigma} - [\Phi(\sigma) + \overline{\Phi(\sigma)}] \tag{24a–d}$$

where in the last two equations we have also used (12), (13) and (19), and the relations

$$l_1 = -\sum_{k=1}^{3} a\mathscr{R}\{c_k\} + ib\mathscr{R}\{c_k\mu_k\}; \quad l_2 = -\sum_{k=1}^{3} a\mathscr{R}\{c_k\mu_k\} + ib\mathscr{R}\{c_k\mu_k^2\};$$

$$l_3 = -\sum_{k=1}^{3} a\mathscr{R}\{c_k\lambda_k\} + ib\mathscr{R}\{c_k\lambda_k\mu_k\}; \quad l_4 = -\sum_{k=1}^{3} a\mathscr{R}\{c_k\kappa_k\} + ib\mathscr{R}\{c_k\kappa_k\mu_k\} \tag{25}$$

A major step towards mathematical simplification is attained if (21)–(23) are used in (24d) to write the coefficients d_j as functions of l_4, a_{kj}, κ_k and ρ_0, that is

* We note that (20) and (21) are the counterparts of (44) and (45) of HS.

$$d_j = \frac{1}{1-\rho_0^{4j}}\left\{\sum_{k=1}^{3}[\rho_0^{2j}\kappa_k a_{kj} - \bar{\kappa}_k \bar{a}_{kj}] + \delta_{j1}(\bar{l}_4 - \rho_0^{2j} l_4)\right\} \tag{26}$$

which, once substituted in (24c) together with (21) in (24a) and (24b), produces a new system of three equations in the unknowns a_{kj}:

$$\sum_{k=1}^{3} a_{kj} = \delta_{j1} l_1\,; \quad \sum_{k=1}^{3} \mu_k a_{kj} = \delta_{j1} l_2\,; \quad \sum_{k=1}^{3} \lambda_{kj} a_{kj} + \bar{\kappa}_{kj}\bar{a}_{kj} = \delta_{j1} l'_3 \tag{27}$$

where δ_{ij} is the Kronecker delta and

$$\lambda_{kj} = \lambda_k - \frac{i\varepsilon_0}{1-\rho_0^{4j}}(1+\rho_0^{4j})\kappa_k\,; \quad \bar{\kappa}_{kj} = \frac{i\varepsilon_0}{1-\rho_0^{4j}}2\rho_0^{2j}\bar{\kappa}_k$$

$$l'_3 = l_3 + \frac{i\varepsilon_0}{1-\rho_0^{4j}}[2\rho_0^{2j}\bar{l}_4 - (1+\rho_0^{4j})l_4] \tag{28}$$

Now, since $\delta_{j1} = 0$ when $j = 2, 3$, the solution of (27) implies that all a_{kj} vanish, provided the 6×6 matrix of the coefficients of a_{kj} and $\bar{a}_{kj}$ is not singular. This condition is equivalent to requiring that the determinant of the 2×2 matrix

$$\begin{Vmatrix} A_j & \bar{B}_j \\ B_j & \bar{A}_j \end{Vmatrix}$$

does not vanish, where the entries are given by

$$A_j = \frac{1}{\mu_2-\mu_1}[(\mu_3-\mu_2)\lambda_{1j} + (\mu_1-\mu_3)\lambda_{2j} + (\mu_2-\mu_1)\lambda_{3j}]$$

$$B_j = \frac{1}{\mu_2-\mu_1}[(\mu_3-\mu_2)\kappa_{1j} + (\mu_1-\mu_3)\kappa_{2j} + (\mu_2-\mu_1)\kappa_{3j}] \tag{29}$$

Consequently, (27) is reduced to values of $j = 1$ only, becoming

$$\sum_{k=1}^{3} a_{k1} = l_1\,; \quad \sum_{k=1}^{3} \mu_k a_{k1} = l_2\,; \quad \sum_{k=1}^{3} \lambda_{k1} a_{k1} + \bar{\kappa}_{k1}\bar{a}_{k1} = l'_3 \tag{30}$$

where λ_{k1} and $\bar{\kappa}_{k1}$ are particular cases of (28). Solving (30) yields

$$a_{11} = \frac{1}{\mu_2-\mu_1}[(\mu_3-\mu_2)a_{31} + \mu_2 l_1 - l_2]$$

$$a_{21} = \frac{1}{\mu_2-\mu_1}[(\mu_1-\mu_3)a_{31} + l_2 - \mu_1 l_1]$$

$$a_{31} = \frac{\bar{A}_1 C_1 - \bar{B}_1 \bar{C}_1}{|A_1|^2 - |B_1|^2} \tag{31a–c}$$

with

$$A_1 = f + (\alpha^{-1}+\alpha)\varepsilon_0 c_{11} g\,;$$

$$B_1 = (\alpha^{-1}-\alpha)\varepsilon_0 c_{11} g\,;$$

$$C_1 = l_3 - (h_1 l_1 + h_2 l_2) - \frac{i\varepsilon_0}{2}(\alpha^{-1}+\alpha)[l_4 - b_{13} l_2 + c_{11}(h_3 l_1 + h_4 l_2)]$$

$$+\frac{i\varepsilon_0}{2}(\alpha^{-1}-\alpha)[\bar{l}_4-b_{13}\bar{l}_2+c_{11}(\bar{h}_3\bar{l}_1+\bar{h}_4\bar{l}_2)] \tag{32a–c}$$

being a consequence of (28) and (29) when $j = 1$, and

$$\alpha = \frac{b}{a}$$

$$f = \frac{1}{\mu_2-\mu_1}[(\mu_3-\mu_2)\lambda_1+(\mu_1-\mu_3)\lambda_2+(\mu_2-\mu_1)\lambda_3]$$

$$g = \frac{-i}{2(\mu_2-\mu_1)}[(\mu_3-\mu_2)\lambda_1\mu_1+(\mu_1-\mu_3)\lambda_2\mu_2+(\mu_2-\mu_1)\lambda_3\mu_3]$$

$$h_1 = \frac{\mu_2\lambda_1-\mu_1\lambda_2}{\mu_2-\mu_1}; \quad h_2 = \frac{\lambda_2-\lambda_1}{\mu_2-\mu_1}$$

$$h_3 = \frac{\mu_1\mu_2(\lambda_2-\lambda_1)}{\mu_2-\mu_1}; \quad h_4 = \frac{\lambda_1\mu_1-\lambda_2\mu_2}{\mu_2-\mu_1} \tag{33}$$

At this point we can verify that the results obtained so far include, as a particular case, those of the impermeable model. Indeed, when $\varepsilon_0 = 0$ we find that

$$A_1 = f; \quad B_1 = 0; \quad C_1 = l_3-(h_1l_1+h_2l_2)$$

which gives

$$a_{k1} = \sum_{j=1}^{3}\Lambda_{kj}l_j$$

where Λ_{kj} and l_j are the coefficients appearing in (50) and (54) of HS.

5. ELECTRO-ELASTIC FIELDS IN THE MATERIAL

Using the results of the previous section the complex potentials reduce to

$$\varphi_k(z_k) = c_kz_k+a_{k0}+\frac{a_{k1}}{\zeta_k} \tag{34}$$

while their derivatives, by virtue of (10), become

$$\varphi_k'(z_k) = c_k-\frac{a_{k1}}{\zeta_k^2\omega_k'(\zeta_k)} \tag{35}$$

which, by means of (2), determine the electro-elastic variables up to the constants c_k that must be found through the boundary conditions when $z_k \to \infty$ according to the following procedure: by virtue of the constitutive equations used in this article, at infinity one can prescribe stress and induction independently of each other, thus generating a system of five equations for the six unknowns involved in the real and imaginary parts of the constants c_k. As noted in HS we can, without loss in generality, set the imaginary part of one of these constants equal to zero. Therefore, as before, we set $\mathscr{I}\{c_1\} = 0$. Hence, under prescribed values of stress $\mathbf{T}^\infty$ and induction $\mathbf{D}^\infty$, use of (2) and (35) when $z_k \to \infty$ yields

$$\mathbf{T}^{\infty} = 2\mathscr{R}\sum_{k=1}^{3}\begin{Bmatrix}\mu_k^2\\ 1\\ -\mu_k\end{Bmatrix}c_k; \quad \mathbf{D}^{\infty} = 2\mathscr{R}\sum_{k=1}^{3}\begin{Bmatrix}\mu_k\\ -1\end{Bmatrix}\lambda_k c_k \tag{36a–b}$$

which once compared with (25) determine the "load parameters"

$$l_1 = \frac{-a}{2}(T_{22}^{\infty}-i\alpha T_{12}^{\infty}); \quad l_2 = \frac{a}{2}(T_{12}^{\infty}-i\alpha T_{11}^{\infty}); \quad l_3 = \frac{a}{2}(D_2^{\infty}-i\alpha D_1^{\infty}) \tag{37}$$

It should be noted that, as a result of the piezoelectric effect, an electric field is induced by the applied load at remote distances in the material, whose value is obtained inserting (36) in (1b) giving

$$\mathbf{E}^{\infty} = 2\mathscr{R}\sum_{k=1}^{3}\begin{Bmatrix}1\\ \mu_k\end{Bmatrix}\kappa_k c_k \tag{38}$$

which through comparison with (25) renders

$$l_4 = -\frac{a}{2}(E_1^{\infty}+i\alpha E_2^{\infty}) \tag{39}$$

That is, a model based on exact boundary conditions produces one more load parameter than the impermeable model. The extra parameter, however, is not prescribed independently, but determined from the constitutive equations. To illustrate this point consider the body subjected to a state of stress and induction of the form $\mathbf{T}^{\infty} = T_0\mathbf{e}_2\otimes\mathbf{e}_2$ and $\mathbf{D}^{\infty} = D_0\mathbf{e}_2$, where $\mathbf{e}_2$ is the unit vector in x_2-direction. As a consequence, strain and electric field are induced at infinity whose components are calculated by means of (1a) and (1b), resulting in: $S_{22}^{\infty} = a_{22}T_0+b_{22}D_0$ and $E_2^{\infty} = -b_{22}T_0+c_{22}D_0$ (all other components being equal to zero). That is, the body will deform not only because of the applied forces but also due to the piezoelectric effect. Similarly, an electric field is not only the expected result from the applied charges but also the consequence of deformations generated by the forces. In this case the load parameters reduce to

$$l_1 = \frac{-aT_0}{2}, \quad l_2 = 0, \quad l_3 = \frac{aD_0}{2}, \quad l_4 = \frac{-ib}{2}(b_{22}T_0+c_{22}D_0)$$

From an experimental point of view, it is much simpler to measure (or impose) an applied voltage (from where we can calculate **E**) than the charge associated with **D**. Therefore, suppose that the loading conditions are given by $\mathbf{E}^{\infty} = E_0\mathbf{e}_2$. Hence, although no forces are applied in this instance, the body will stretch or contract due to piezoelectricity in the amount $S_{22}^{\infty} = b_{22}E_0/c_{22}$, and the load parameters become

$$l_1 = 0, \quad l_2 = 0, \quad l_3 = \frac{aE_0}{2c_{22}}, \quad l_4 = \frac{-ibE_0}{2}$$

Finally, suppose that we stretch the body according to $\mathbf{S}^{\infty} = S_0\mathbf{e}_2\otimes\mathbf{e}_2$. As a result, stresses are generated with values given by: $T_{11}^{\infty} = -a_{12}S_0/(a_{11}a_{22}-a_{12}^2)$, $T_{22}^{\infty} = a_{22}S_0/(a_{11}a_{22}-a_{12}^2)$ and $T_{12}^{\infty} = 0$. Moreover, through electromechanical interaction, an electric field is induced with components $E_1^{\infty} = 0$ and $E_2^{\infty} = (b_{21}a_{12}-b_{22}a_{22})S_0/(a_{11}a_{22}-a_{12}^2)$. Hence, in this case

$$l_1 = \frac{-aa_{22}S_0}{2(a_{11}a_{22}-a_{12}^2)}, \quad l_2 = \frac{iba_{12}S_0}{2(a_{11}a_{22}-a_{12}^2)}, \quad l_3 = 0, \quad l_4 = \frac{-ib(b_{21}a_{12}-b_{22}a_{22})S_0}{2(a_{11}a_{22}-a_{12}^2)}$$

Naturally, combinations of $\mathbf{S}^\infty$ and $\mathbf{D}^\infty$, $\mathbf{S}^\infty$ and $\mathbf{E}^\infty$ or $\mathbf{T}^\infty$ and $\mathbf{E}^\infty$ can be worked out in the same fashion.

Knowledge of the parameters l_j implies full knowledge of the constants c_k, which, in turn, allows us to fully determine any of the elastic and electric variables existing in matter. In particular,

$$\mathbf{T} = \mathbf{T}^\infty + 2\mathscr{R}\sum_{k=1}^{3}\begin{Bmatrix}\mu_k^2\\1\\-\mu_k\end{Bmatrix}\mathscr{Z}_k$$

$$\mathbf{D} = \mathbf{D}^\infty + 2\mathscr{R}\sum_{k=1}^{3}\begin{Bmatrix}\mu_k\\-1\end{Bmatrix}\lambda_k\mathscr{Z}_k \quad \text{in}\,\Omega$$

$$\mathbf{E} = \mathbf{E}^\infty + 2\mathscr{R}\sum_{k=1}^{3}\begin{Bmatrix}1\\\mu_k\end{Bmatrix}\kappa_k\mathscr{Z}_k \tag{40a–c}$$

where

$$\mathscr{Z}_k = \mathscr{Z}_k(\mu_k, z_k) = \frac{-a_{k1}}{\zeta_k^2\omega_k'(\zeta_k)} \tag{41}$$

At this time it seems worthy to recapitulate the steps one needs to follow to find exact expressions for the mechanical and electrical variables in Ω. In a given problem, the input consists of material properties, geometry of the ellipse, and loading conditions. The steps to follow are: (a) calculate μ_k, λ_k, κ_k, α and l_k. (b) Calculate A_1, B_1 and C_1 using (32), where we should notice that the first two are independent of the load. (c) Calculate a_{11}, a_{21} and a_{31} using (31). (d) Substitute in (40) the results from the previous steps.

For practical purposes we also review the corresponding units of the various coefficients involved in the present analysis. In the international system of units (with volts V, meters m and Newtons N as fundamental units) they are: $[A_1] = [B_1] = \text{m/V}$, $[C_1] = \text{N/V}$, $[l_1] = [l_2] = \text{N/m}$, $[l_3] = \text{N/V}$, $[l_4] = \text{V}$, $[\lambda_k] = \text{m/V}$, $[\kappa_k] = \text{Vm/N}$, $[a_{k1}] = \text{N/m}$ and μ_k non dimensional.

The expressions given by (40) are in terms of Cartesian coordinates x_1 and x_2 and are valid everywhere in Ω. Of particular interest are their expressions on Γ itself in terms of components normal and tangent to the curve. This is achieved by introducing the following change of variables:

$$z_k = a\cos\theta + \mu_k b\sin\theta \tag{42}$$

where $0 \leqslant \theta < 2\pi$ is measured along Γ in counter clockwise sense. As a result, (41) becomes

$$\mathscr{Z}_k = \mathscr{Z}_k(\mu_k, \theta) = \frac{a_{k1}(\sin\theta + i\cos\theta)}{a\sin\theta - \mu_k b\cos\theta} \tag{43}$$

In the Cartesian basis, the unit normal and tangent vectors to Γ are given by $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ and $\mathbf{s} = -n_2\mathbf{e}_1 + n_1\mathbf{e}_2$ where the components of $\mathbf{n}$ are obtained from (11) and (17) with $\zeta = e^{i\theta}$, yielding

$$n_1 + in_2 = \frac{\alpha\cos\theta + i\sin\theta}{\Theta}; \quad \Theta = \sqrt{\alpha^2\cos^2\theta + \sin^2\theta} \tag{44}$$

On Γ the stress vector vanishes, thus $\mathbf{Tn} = 0$, or equivalently, $T_{nn} = T_{ns} = 0$. However, a hoop stress can in general be induced, whose value is given by

$$T_{ss}(\theta) = \frac{1}{\Theta^2}[T^{\infty}_{11}\sin^2\theta + T^{\infty}_{22}\alpha^2\cos^2\theta - 2T^{\infty}_{12}\alpha\cos\theta\sin\theta] + \frac{2}{\Theta^2}\mathscr{R}\sum_{k=1}^{3}(\mu_k\sin\theta + \alpha\cos\theta)^2\mathscr{Z}_k \tag{45}$$

A discussion concerning the behavior of this stress component for various values of α is relegated to the last section of the article.

Next, we turn our attention to the expressions for the electric variables on the curve Γ. Of particular interest are the normal component of the induction and the tangential component of the field. Using (40b) and (40c) one obtains

$$D_n(\theta) = \frac{2}{a\Theta}\mathscr{R}\left\{ie^{-i\theta}\left[l_3 - \sum_{k=1}^{3}\lambda_k a_{k1}\right]\right\} \tag{46a}$$

and

$$E_s(\theta) = \frac{2}{a\Theta}\mathscr{R}\left\{ie^{-i\theta}\left[l_4 - \sum_{k=1}^{3}\kappa_k a_{k1}\right]\right\} \tag{46b}$$

Equation (46b) is crucial to study the behavior of the field at the tip of a crack, which is one of the main themes behind the motivation of this paper. Hence, consider the case when the body is subjected to a load $\mathbf{E}^{\infty} = E_0\mathbf{e}_2$ as a result, it can be shown that using (31)–(33) one obtains

$$\sum_{k=1}^{3}\kappa_k a_{k1} = 2gc_{11}a_{31}i, \quad C_1 = \frac{a}{2}(c_{22}^{-1} - \varepsilon_0)E_0, \quad a_{31} = \frac{a(c_{22}^{-1} - \varepsilon_0)\alpha E_0}{2g(\alpha\gamma + 2\varepsilon_0 c_{11})}$$

where $\gamma = f/g$ is a real, positive number (typically in the range $1 < \gamma < 2$). Thus, substituting these results and (44) in (46b) yields

$$E_s(\theta) = \frac{(\alpha\gamma + 2c_{11}c_{22}^{-1})E_0\alpha\cos\theta}{(\alpha\gamma + 2\varepsilon_0 c_{11})\Theta} \tag{47}$$

Now it is clear that, if the analysis of the defect problem were based on the impermeable model (i.e. when $\varepsilon_0 = 0$), the field would become singular when $\alpha \to 0$ at the tip of the crack (when $\theta = 0$). However, if ε_0 is retained, in the limit of the sharp crack we have

$$\lim_{\alpha\to 0} E_s(0) = \frac{E_0}{\varepsilon_0 c_{22}} \tag{48}$$

That is, the field is bounded as predicted by McMeeking (1989) and Dunn (1994) through alternative models. Nevertheless, from a practical point of view, this result offers small consolation. Indeed, since for most ceramics $\varepsilon_0 c_{22} \sim 10^{-4}$, it is clear that applied fields of only 100 V/m can produce depoling of the material and, therefore, loss of the piezoelectric effect.

Since it is most likely that in real situations $\alpha \neq 0$, the natural question coming to mind is: what are the limitations of the impermeable model so its results can be used with confidence? The answer is provided by (47). Keeping in mind that $\alpha \leqslant 1$, $1 < \gamma < 2$, and that for most ceramics $c_{11}c_{22}^{-1} \sim 1$ and $\varepsilon_0 c_{11} \sim 10^{-4}$, the following can be postulated: (1) if $\alpha \gg \varepsilon_0 c_{11}$, then the impermeable and exact models provide virtually the same results. For example, a slender ellipse with $\alpha = 10^{-2}$ could be modelled reasonably well with the condition $\varepsilon_0 = 0$. (2) If $\alpha \sim \varepsilon_0 c_{11}$, then there are two competing factors to take into account in the denominator of (47) and the limits $\varepsilon \to 0$ and $\alpha \to 0$ should be taken in either sequence to find the dramatic differences provided by both models as discussed before.

6. ELECTRIC FIELDS IN THE CAVITY

By using (12), (22) and (23) with $j = 1$, the expression for the electric potential inside the cavity can be written as follows:

$$\phi^c(x_1, x_2) = (d_0 + \bar{d}_0) + \left(\frac{\rho_0^2}{\zeta} + \zeta\right)d_1 + \left(\frac{\rho_0^2}{\bar{\zeta}} + \bar{\zeta}\right)\bar{d}_1$$

which after inverting (11) reduces to

$$\phi^c(x_1, x_2) = (d_0 + \bar{d}_0) + \frac{2}{a+b}[z d_1 + \bar{z}\bar{d}_1]$$

from where, in turn, we can determine the components of the electric field, namely

$$E_1^c = -\frac{2(d_1 + \bar{d}_1)}{a+b}$$

$$E_2^c = -\frac{2i(d_1 - \bar{d}_1)}{a+b}$$

indicating that the electric field is uniform inside the hole. The quantities in parentheses can be found explicitly, they are given by

$$d_1 + \bar{d}_1 = \frac{-2}{1+\rho_0^2}\left[\sum_{k=1}^{3} \mathscr{R}\{\kappa_k a_{k1}\} - \mathscr{R}\{l_4\}\right]$$

$$d_1 - \bar{d}_1 = \frac{2i}{1-\rho_0^2}\left[\sum_{k=1}^{3} \mathscr{I}\{\kappa_k a_{k1}\} - \mathscr{I}\{l_4\}\right]$$

Thus the electric field components in Ω_c become

$$E_1^c = E_1^\infty + \frac{2}{a}\sum_{k=1}^{3} \mathscr{R}\{\kappa_k a_{k1}\}$$

$$E_2^c = E_2^\infty + \frac{2}{b}\sum_{k=1}^{3} \mathscr{I}\{\kappa_k a_{k1}\} \qquad (49a\text{–}b)$$

Verification of the conditions of continuity regarding the normal and tangential components of **D** and **E**, respectively, is another valid test to check the correctness of our results. In Section 5 we found these components by approaching the boundary from Ω. The same components calculated by approaching Γ from Ω_c are given by

$$D_n^c = \frac{2\varepsilon_0}{a\Theta}\mathscr{R}\{il_3 e^{-i\theta}\} + \frac{2\varepsilon_0}{a\Theta}\sum_{k=1}^{3}\mathscr{R}\{(\alpha\cos\theta - i\alpha^{-1}\sin\theta)\kappa_k a_{k1}\} \tag{50a}$$

and

$$E_s^c = \frac{2}{a\Theta}\mathscr{R}\left\{ie^{-i\theta}\left[l_4 - \sum_{k=1}^{3}\kappa_k a_{k1}\right]\right\} \tag{50b}$$

Equations (46b) and (50b) show that $E_s^c = E_s$. To verify that $D_n^c = D_n$ one must use (24c) in (46a). Notice also that $D_n^c = 0$ when $\varepsilon_0 = 0$.

The characteristics of the electric field within the void are illustrated by means of two examples sharing the following result:

$$\sum_{k=1}^{3}\kappa_k a_{k1} = 2gc_{11}a_{31}i - c_{11}h_3 l_1. \tag{51}$$

Example 1: Suppose that the body is subjected to voltages in x_1- and x_2-directions, that is, the applied load is given by $\mathbf{E}^\infty = E_1^\infty \mathbf{e}_1 + E_2^\infty \mathbf{e}_2$. The load parameters become

$$l_1 = l_2 = 0, \quad l_3 = \frac{a}{2c_{11}}\left(\frac{c_{11}}{c_{22}}E_2^\infty - i\alpha E_1^\infty\right), \quad l_4 = \frac{-a}{2}(E_1^\infty + i\alpha E_2^\infty)$$

and, by means of (32c),

$$C_1 = \frac{a}{2}(c_{22}^{-1} - \varepsilon_0)E_2^\infty + \frac{ib}{2}(\varepsilon_0 - c_{11}^{-1})E_1^\infty$$

Using this expression together with (32a–b) in (31c) gives

$$a_{31} = \frac{a(c_{22}^{-1} - \varepsilon_0)\alpha E_2^\infty}{2g(\alpha\gamma + 2\varepsilon_0 c_{11})} + i\frac{b(\varepsilon_0 - c_{11}^{-1})E_1^\infty}{2g(\gamma + 2\alpha\varepsilon_0 c_{11})} \tag{52}$$

Further algebraic manipulations with (51) yields

$$\sum_{k=1}^{3}\kappa_k a_{k1} = \frac{(1 - \varepsilon_0 c_{11})bE_1^\infty}{(\gamma + 2\alpha\varepsilon_0 c_{11})} + i\frac{c_{11}(c_{22}^{-1} - \varepsilon_0)\alpha a E_2^\infty}{\alpha\gamma + 2\varepsilon_0 c_{11}}$$

where $\gamma = f/g$ as before. Thus, extracting its real and imaginary parts one obtains

$$E_1^c(x_1, x_2) = \frac{(\gamma + 2\alpha)E_1^\infty}{\gamma + 2\alpha\varepsilon_0 c_{11}}$$

$$E_2^c(x_1, x_2) = \frac{(\alpha\gamma + 2c_{11}c_{22}^{-1})E_2^\infty}{\alpha\gamma + 2\varepsilon_0 c_{11}} \tag{53a–b}$$

First, we note that the case of the impermeable hole is obtained by setting $\varepsilon_0 = 0$. Under such circumstance, it is clear that a field applied in x_1-direction is not perturbed by a crack, while a voltage applied in x_2-direction yields a field singular everywhere within the crack. Second, in the presence of a crack, E_2^∞ induces a field given by (48) since from continuity $E_s(0) = E_2^c(a, 0)$. Third, for aspect ratios α smaller than one but still larger than $\varepsilon_0 c_{11}$ we deduce as before, that the results emanating from an exact or impermeable model yield virtually the same results.

Example 2: Assume the body subject to a state of stress given by $\mathbf{T}^{\infty} = T_0\mathbf{e}_2 \otimes \mathbf{e}_2$, in which case the loading parameters become

$$l_1 = -\frac{aT_0}{2}, \quad l_2 = 0, \quad l_3 = \frac{ab_{22}T_0}{2c_{22}}, \quad l_4 = 0$$

and

$$C_1 = l_3 - h_1 l_1 + \frac{i\varepsilon_0}{2}c_{11}[(\alpha^{-1} - \alpha)\bar{h}_3 - (\alpha^{-1} + \alpha)h_3]l_1$$

while

$$a_{31} = \frac{[\alpha(l_3/l_1) - \alpha h_1^R + \varepsilon_0 c_{11} h_3^I]l_1}{(\alpha\gamma + 2\varepsilon_0 c_{11})g} - i\frac{[h_1^I + \varepsilon_0 c_{11} h_3^R \alpha]l_1}{(\gamma + 2\alpha\varepsilon_0 c_{11})g}$$

where $h_1^R = \mathscr{R}\{h_1\}, h_1^I = \mathscr{I}\{\mathrm{h}_1\}$, with similar definitions for h_3. Therefore, using (51)

$$\sum_{k=1}^{3} \kappa_k a_{k1} = \frac{c_{11}[2h_1^I - \gamma h_3^R]l_1}{\gamma + 2\alpha\varepsilon_0 c_{11}} + \frac{ic_{11}[2\alpha(l_3/l_1) - 2\alpha h_1^R - \alpha\gamma h_3^I]l_1}{\alpha\gamma + 2\varepsilon_0 c_{11}}$$

which produces

$$E_1^c(x_1, x_2) = \frac{c_{11}[\gamma h_3^R - 2h_1^I]T_0}{\gamma + 2\alpha\varepsilon_0 c_{11}}$$

$$E_2^c(x_1, x_2) = \frac{c_{11}[2(b_{22}/c_{22}) + 2h_1^R + \gamma h_3^I]T_0}{\alpha\gamma + 2\varepsilon_0 c_{11}} \qquad \text{(54a–b)}$$

That is, a state of stress in x_2-direction induces a field with components in both x_1- and x_2-directions. These components, however, not only are quite different in terms of their magnitudes (for fixed values of α and γ $E_1^c \ll E_2^c$) but also with respect to their behavior when the hole becomes a slit crack under the conditions of impermeability. Clearly, E_1^c is indifferent to the nature of the boundary conditions when $\alpha = 0$. The same cannot be said about E_2^c, which remains bounded in the case of a slit crack when exact electric boundary conditions are enforced, and becomes singular if analyzed through the impermeable model.

7. ON STRESSES INDUCED BY ELECTRIC FIELDS

One of the most interesting aspects concerning piezoelectric materials with cracks has to do with the effects the electric field has on the stress distribution around a crack tip. We note that (at least from a qualitative point of view), the fracture characteristics of cracked piezoelectric solids subjected to purely mechanical loads can be described by fracture mechanics concepts of anisotropic media (Sosa and Pak, 1990). That is, the asymptotic expressions for stresses have the classical singularity $1/\sqrt{r}$ at the tip of the crack and the functions reflecting the angular distributions are also functions of the material properties.

The case of a piezoelectric body subjected, in addition to mechanical forces, to remote electric field has consequences that we would like to investigate briefly in this section. It is interesting to note that some experimental observations and analytical models have speculated that an electric field may enhance or retard crack propagation initiated by the application of mechanical forces. In this section we study, therefore, the case of the piezoelectric solid subjected to a field in x_2-direction and draw our attention to the behavior of the stress component T_{ss} on Γ at the point $\theta = 0$. Thus, from (45) we have

$$T_{ss}(0) = T_{22}(a,0) = \frac{2}{b}\sum_{k=1}^{3}\mathscr{R}\left\{-i\frac{a_{k1}}{\mu_k}\right\} \tag{55}$$

where by means of (31)

$$-i\sum_{k=1}^{3}\frac{a_{k1}}{\mu_k} = \left[-i\frac{(\mu_3-\mu_2)\mu_1^{-1}+(\mu_1-\mu_3)\mu_2^{-1}+(\mu_2-\mu_1)\mu_3^{-1}}{\mu_2-\mu_1}\right]a_{31} \tag{56}$$

where a_{31} is given by (52) with $E_1^\infty = 0$ and $E_2^\infty = E_0$. Furthermore, we can show that (56) is real and can be written as

$$-i\sum_{k=1}^{3}\frac{a_{k1}}{\mu_k} = \frac{aK(c_{22}^{-1}-\varepsilon_0)\alpha E_0}{2(\alpha\gamma+2\varepsilon_0 c_{11})}$$

where the number $K = [.]/g$ is real and positive, and [.] represents the quantity in brackets of (56). Therefore (55) becomes

$$T_{ss}(0) = \frac{K(c_{22}^{-1}-\varepsilon_0)E_0}{\alpha\gamma+2\varepsilon_0 c_{11}} \tag{57}$$

If the piezoelectric solid is, in addition, subjected to remote forces, the corresponding stresses can be added to (57) according to the principle of superposition. On this point, it is useful to note that the sign of $T_{ss}(0)$ in (57) depends solely on the sign of E_0, since the rest of the expression is positive. Thus, the electric field can increase or reduce the intensity of the normal stress generated by say tensile forces applied in an independent manner. Due to the implications of this effect it seems appropriate to investigate the order of magnitude of the stress induced only by the electric field. To this end, we note that for piezoelectric ceramics, the quantities involved in (57) have the following orders of magnitude: c_{11}, $c_{22} \sim 10^8$, $\varepsilon_0 \sim 10^{-12}$, $\gamma \sim 1$ and $K \sim 10^8$–10^9. Hence, according to these values, the stress induced by the field for aspect ratios in the range, say, $10^{-2} < \alpha < 1$ is given (in its most critical condition) by

$$T_{22}(0) \sim \frac{10\times E_0}{\alpha} \tag{58}$$

That is, the more slender the ellipse, the larger the stress in accordance to physical intuition. In some cases the value given by (58) may have a substantial incidence in the overall value of the normal stress. For example, a field of 10^5 V/m applied alone could induce a stress of up to 100 MPa if the axes of the ellipse are in the ratio 1/100. Finally, notice that within the framework of the impermeable model the stress is singular at the tip of a crack, while bounded if ε_0 is retained.

With the expressions provided in the previous sections we could generate other interesting results. We refrain, however, of such exercises since they were not at the core of our objectives.

8. CLOSURE

The problem of an elliptic hole embedded in a transversely isotropic piezoelectric solid has been addressed within the framework of in-plane electro-elastic interactions. Contrary to what has been common in the literature, exact electric boundary conditions have been enforced at the rim of the hole. As a consequence, expressions for the electric variables are provided not only in the material but within the hole as well. It has been shown that, as predicted by other authors within simpler configurations, invoking the condition of electric impermeability at the boundary of the cavity may result into erroneous conclusions, which become particularly relevant for the case of very slender ellipses or sharp cracks, where the approximate theory predicts singular fields. When the domain of the hole is also taken into consideration, it is seen that the fields at a crack tip are certainly large but bounded by the permittivities of the material and gas enclosed by the hole. It should be emphasized that despite the constraints of the model, regarding the orientation of the defect, the model contemplates the most general coupling effects and material anisotropy. Furthermore, despite the mathematical complexities inherent to these problems, the exact expressions here provided are strikingly simple in form and are ready to be utilized in conjunction with numerical and symbolic algorithms.

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